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Robust Control Design Strategy with Parameter-Dominated Uncertainty

Moshe Idan* and Guy E. Shaviv†
Technion—Israel Institute of Technology, Haifa 32000, Israel

A robust controller design strategy is presented for systems, the parameters of which are affected dominantly by a few unknown physical quantities. The use of the functional dependence of the model parameters on these physical quantities results in an accurate description of the model uncertainty space. This potentially may lead to a better tradeoff between performance and robustness. The proposed procedure for modeling such uncertain systems relies on the linear–fractional–transformation algebra and reduction of the uncertainty block dimension. The resulting model can be used for control synthesis using any robust control design technique. In this work μ synthesis was adopted since it best fits the structured uncertainty model obtained. An example presents a controller design for an unmanned flight vehicle, the model parameters of which depend on the unknown mass of its payload. The aircraft equations are derived, and a μ controller is synthesized. The controller is compared to other control methods, and its advantages are demonstrated.

I. Introduction

DYNAMIC systems are often described by linear models with numerous parameters depending predominantly on a few unknown physical quantities. As an example, parameters of a linear model of an aircraft strongly depend on the location of vehicle center of gravity c.g., air density ρ, trim angle of attack α_{trim}, and Mach number M, as well as other parameters. Each of these parameters changes some or all of the model coefficients simultaneously in a known fashion.[1] Some of these dependencies may be modeled analytically in a relatively simple form (e.g., the model coefficient dependence on the aircraft c.g. location or on the air density) whereas others may be more difficult to model analytically, but still could be expressed empirically based on experimental results (e.g., model dependence on the Mach number around $M = 1$ or on the angle of attack for large α_{trim}). In addition, there are model uncertainties that are not related to any common physical quantity, such as uncertainties in the nondimensional aerodynamic stability and control derivatives. In this work it is assumed that the overall effect of these uncertainties is small compared to those caused by the uncertainties in the aforementioned physical parameters. The design goal is to achieve robust performance for the uncertainties in these parameters.

A related problem can be formulated when designing a controller for a dynamic system operating around several equilibrium points distinguished by different values of some physical parameters. These parameters can be considered unknown and allowed to perturb within a specified range. The requirement for the controller is then to provide robust stability and performance of the system while the physical parameters vary in the given range.

Several approaches for designing robust controllers in the presence of parametric uncertainty have been introduced in the past. Reference 2 reviews a class of methods based on extensions of the Kharitonov theorem. In these methods the plant uncertainty is described as uncertainties in the coefficients of the closed-loop characteristic polynomial. The drawback of these methods is in the difficulty to relate accurately the physical plant uncertainties to variations in the closed-loop characteristic polynomial coefficients. The multimodel approach[3,4] evaluates the system model at several operating points and attempts to find a controller that meets performance criteria (defined as Γ stability) for all of the models simultaneously. This approach leads to a complex design problem when the number of controller and uncertainty parameters increases. In addition, there is no guarantee on the performance at operating points not considered in the design. The quantitative feedback theory[5] attempts to find the regions, named templates, on the Nichols chart that encompasses all of the possible frequency responses of the uncertain plant. No general method exists for constructing these templates, which are calculated for a finite number of frequencies only.

These are but some of the methods that tackle the problem of real parametric uncertainties. The shortcoming of these methods is the difficulty in obtaining an accurate description of real parameter model uncertainties while retaining their functional dependence on a few uncertain physical quantities. An inaccurate uncertainty space description may result in undesirable overbounding of the physical uncertainty space. This led to the need for a different uncertainty modeling approach.

In this paper robust control design methodology for systems with model uncertainties dominated by a few physical parameters is presented. The strategy is to incorporate the known functional dependence of the model coefficients on these few uncertain physical parameters into the uncertainty model, so that the undesirable uncertainty overbounding is avoided. The use of this knowledge may also reduce the controller complexity by reducing the dimension of the uncertainty block and may aid in achieving a better compromise between performance and robustness goals. A similar modeling approach was presented in Refs. 6 and 7; however, only special cases of parameter dependencies were solved. This paper presents a procedure for a more general case of real parameter uncertainties. Although there is no guarantee that the order of the uncertainty block is minimal, the proposed method can model exactly any rational functional dependence without overbounding.

In the next section the modeling approach is presented. The linear fractional transformations (LFTs) are briefly reviewed in Sec. III. In Sec. IV, as an example, the short-term dynamics of an unmanned flight vehicle and its dependence on the uncertainty in the payload weight is modeled, followed by a μ controller synthesis in Sec. V. Comparisons with controllers designed using models with general bounds on the uncertain parameters are presented in Sec. VI. Summary and concluding remarks are given in Sec. VII.

II. Approach

In this section an overview of the main methodology and ideas used in developing the uncertainty model are presented.

When designing robust controllers it is common to have a system as in Fig. 1, where the uncertainty is looped around the plant. Many software packages[8,9] use this structure as the input to the design

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*Annie and Charles Corrin Academic Lecturer, Department of Aerospace Engineering. Member AIAA.

†Graduate Student, Department of Aerospace Engineering.

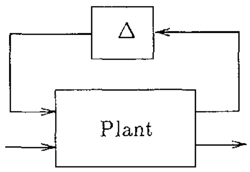

Fig. 1 General block structure with uncertainty.

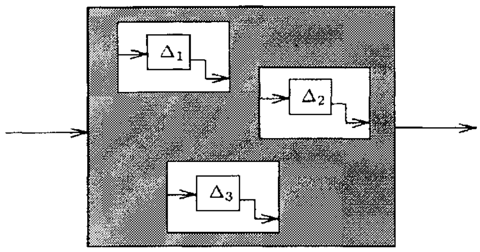

Fig. 2 Uncertainties Δ_i inside the plant model.

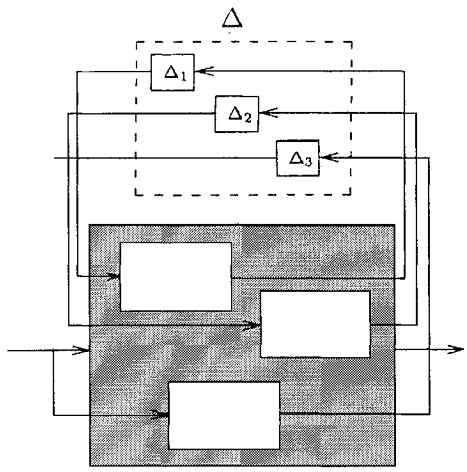

Fig. 3 Uncertainties Δ_i combined into one block (pulled out).

routines. In the process of system modeling, the relations between each of the model parameters to some uncertainty parameters are known[10] and thus are better depicted by the scheme shown in Fig. 2.

To use the available computer software, it is necessary to transform the model from the representation of Fig. 2 to that of Fig. 1, or in other words to pull out the Δ_i, as illustrated in Fig. 3. The process of pulling out the Δ_i creates an uncertainty block Δ whose dimension equals to the number of unknown model parameters.

When the uncertain model parameters are known functions of an unknown physical parameter δ, i.e., $\Delta_i = f_i(\delta)$, treating each Δ_i independently may result in an overbounded uncertainty model. Using the known functional dependencies $f_i(\delta)$, the Δ block can often be transformed into a $\Delta = \delta I_n$ form with only one δ changing independently, where I_n is an $n \times n$ identity matrix. Existence of such transformation is not guaranteed, except for special cases, such as when f_i are rational functions of δ, as discussed in the next section. The advantage of using this repeated scalar block structure is that it describes the model uncertainty accurately. Similar transformations can be performed when the model parameters depend on a few physical parameters.

The dimension n of this repeated scalar block may be large. As will be shown in the sequel, however, this dimension can be often reduced to produce an uncertainty block with a dimension even smaller than the number of the unknown model parameters. This is advantageous from the design standpoint, because a smaller uncertainty block Δ could reduce the controller dimension and potentially its complexity.

The modeling approach presented in this work relies on the general linear feedback loop structure called the LFT. The strategy is to express the functional dependence of the model parameters on the uncertain physical quantities as LFTs and then, using LFT arithmetics, merge these uncertainties into a single block. The dimension of this block, which contains only the physical parameters, is usually large and often can be reduced. The resulting model with the reduced-order uncertainty block is then used to design a robust controller.

This procedure differs from the approach taken in Refs. 6 and 7 in that here the uncertainties are introduced as LFTs acting on the various model parameters. The uncertainty separation process (pulling out the Δ_i) is performed using LFT algebra followed by an uncertainty block reduction procedure. In contrast, Refs. 6 and 7 first separate the plant model into a sum of the nominal and uncertainty dependent parts. The problem then is to express the latter as an LFT with an uncertainty block of minimum order. This problem was solved only for a few special cases, in particular when the uncertainty dependent part is a multilinear function of the uncertainties. Although the procedure suggested here does not guarantee, in general, a model with a minimum-order uncertainty block, it will provide an accurate uncertainty model for any rational uncertainty dependence.

A brief overview of the basic characteristics of LFTs and a procedure for reducing the dimension of LFTs are presented in the next section.

III. Linear Fractional Transformation

A. Overview of LFTs

Let M be a complex matrix partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{C}^{(p_1+p_2)\times(q_1+q_2)} \tag{1}$$

and also let $\Delta_u \in \mathbb{C}^{q_1 \times p_1}$ and $\Delta_l \in \mathbb{C}^{q_2 \times p_2}$. Then the upper and lower LFTs are defined as

$$\mathcal{F}_u(M, \Delta_u) = M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12} \tag{2}$$

and

$$\mathcal{F}_l(M, \Delta_l) = M_{11} + M_{12}\Delta_l(I - M_{22}\Delta_l)^{-1}M_{21} \tag{3}$$

respectively. The existence of the inverses in these equations is necessary for the LFTs to be well posed. The LFT formulas arise naturally when describing feedback systems as in Fig. 4. The resulting closed-loop transfer functions from w to z are $\mathcal{F}_u(M, \Delta_u)$ and $\mathcal{F}_l(M, \Delta_l)$, respectively.

LFTs can be used to describe transfer functions. Let

$$\begin{Bmatrix} \dot{x}(t) \\ y(t) \end{Bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} x(t) \\ u(t) \end{Bmatrix} \tag{4}$$

be a state–space realization of a dynamic system, with real A, B, C, and D matrices of appropriate dimensions. The transfer function from $\boldsymbol{u}(s)$ to $\boldsymbol{y}(s)$

$$\boldsymbol{y}(s) = [D + C(sI - A)^{-1}B]\boldsymbol{u}(s) = T_{yu}(s)\boldsymbol{u}(s) \tag{5}$$

can be described as an LFT:

$$T_{yu}(s) = \mathcal{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \frac{1}{s}I\right) \tag{6}$$

The convenience of LFTs is that many algebraic operations on LFTs result in an LFT. Consequently, a rational function of an LFT can be expressed as an LFT. For example, assume that $q = f(p)$ and $p = \mathcal{F}_u(M, \delta)$ with a scalar uncertainty block δ. If the function $f(\cdot)$ is rational, q can be expressed as

$$q = f(p) = f[\mathcal{F}_u(M, \delta)] = \mathcal{F}_u(M_c, \Delta_c) \tag{7}$$

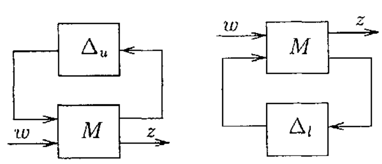

Fig. 4 LFTs represented as linear feedback loops.

for some matrix M_c. Δ_c has the form $\Delta_c = \delta I_c$, where I_c is an identity matrix of some dimension. M_c and Δ_c can be found using LFT arithmetics, described in Ref. 11.

This result will be used to express the algebraic relations between model coefficients and uncertainty parameters as an LFT. In this work, a Mathematica® package was developed to handle the LFT arithmetics symbolically.

B. Reducing the Dimension of LFTs

In modeling uncertain systems, while applying LFT algebra, or when resolving functional expressions such as in Eq. (7), the dimension of the uncertainty block increases. As in manipulation of dynamic systems represented in state space, such an increase may result from the introduction of unnecessary modes into the model, or, in the LFT sense, the introduction of unnecessary repetitions of the scalar uncertainties (δ). It is, therefore, required to reduce the dimension of the Δ block to minimum.

In this work it is assumed that each δ in the uncertainty block has a physical meaning. Thus, the minimization procedure presented here attempts to minimize the repetitive scalar blocks of each uncertainty only. No attempt is made to aggregate several uncertainties and thus obtain a smaller block dimension by using combinations of δ. In addition, the procedure will not attempt to cover the uncertainty space with fewer nonphysical parameters since this could degrade modeling accuracy by distorting the original uncertainty space.

Since the uncertainty block Δ in this work is diagonal, it can be thought of as representing a multidimensional frequency structure. In this sense, the LFT system can be viewed as representing a multidimensional (MD) system where the δ_i are the different transform variables.[12] Let the frequency/uncertainty block structure Δ have the form

$$\Delta = \mathrm{diag}\{\delta_1 I_{q_1}, \ldots, \delta_r I_{q_r}\} \tag{8}$$

where δ_i are the transform variables/uncertainty parameters, each one repeated q_i times. I_{q_i} is a $q_i \times q_i$ identity matrix. Also let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

be a matrix with A, B, and C partitioned compatibly with the block structure Δ as

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & \ddots & \vdots \\ A_{r1} & \cdots & A_{rr} \end{bmatrix}; \qquad B = \begin{bmatrix} B_1 \\ \vdots \\ B_r \end{bmatrix}$$

$$C = [\, C_1 \quad \cdots \quad C_r \,]$$

The input/output for this MD system is given by

$$y = \mathcal{F}_u(M, \Delta)u$$

where

$$\mathcal{F}_u(M, \Delta) = D + C\Delta(I - A\Delta)^{-1}B$$

For any nonsingular matrix T (Ref. 12),

$$\mathcal{F}_u(M, \Delta) = \mathcal{F}_u\left(\begin{bmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{bmatrix}, T\Delta T^{-1}\right) \tag{9}$$

In particular, if T has the structure $T = \mathrm{diag}\{T_1, \ldots, T_r\}$ with $\dim T_i = q_i$, then Δ and T commute, $T\Delta = \Delta T$, and

$$\mathcal{F}_u(M, \Delta) = \mathcal{F}_u\left(\begin{bmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{bmatrix}, \Delta\right) \tag{10}$$

This result can be used to reduce the dimension of the realization (A, B, C, D) with respect to the transform variables δ_i in a similar manner to the balanced truncation procedure presented in Ref. 12. A difficulty arises from the restriction that the transformation matrix T has the block structure of Δ. Thus in this work, the reduction of the LFT dimension is performed for each block separately.

To demonstrate this procedure to reduce the system with respect to δ_1, the matrix M is repartitioned as

$$M = \begin{bmatrix} A_{11} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$$

with

$$\hat{B} = [\, A_{12} \quad \cdots \quad A_{1r} \quad B_1 \,]; \qquad \hat{C} = \begin{bmatrix} A_{21} \\ \vdots \\ A_{r1} \\ C_1 \end{bmatrix}$$

$$\hat{D} = \begin{bmatrix} A_{22} & \cdots & A_{2r} & B_2 \\ \vdots & \ddots & \vdots & \vdots \\ A_{r2} & \cdots & A_{rr} & B_r \\ C_1 & \cdots & C_r & D \end{bmatrix}$$

T_1 is found to obtain a canonical realization of $(A_{11}, \hat{B}, \hat{C}, \hat{D})$, essentially treating the other δ_i loops as inputs/outputs of this system. In this new realization, modes that are uncontrollable and/or unobservable with respect to the transform variable δ_1 can be canceled, and the size of the corresponding block $\delta_1 I_{q_1}$ can be reduced accordingly. This cancelation procedure can then be repeated for the other $\delta_i I_{q_i}$ blocks. In general, it is difficult to ensure that the final realization is minimal, except for special cases, such as those in Refs. 6 and 7. This procedure for reducing the dimension of LFTs is automated in a Mathematica symbolic package developed in this work.

IV. Aircraft Model

An uncertain model of a remotely piloted vehicle (RPV) carrying a payload with an unknown mass is derived in this section using the proposed methodology. The same formulation can be used to design a single controller for payloads with different masses. A dynamic linear model of the RPV with a nominal mass payload is obtained by determining a trimmed flight condition for the nonlinear aircraft equations of motion and linearization of these equations around the trim condition. Only the short-term longitudinal dynamics of the aircraft are treated in this example (the short-period model), although more complex models can be handled in the same way using the approach proposed in this work.

The change in the payload mass affects the model in two ways: it changes the total mass of the aircraft and also shifts the location of its center of gravity (c.g.). In this work it is assumed that the trim velocity is kept the same for all payloads. The modeling approach is to first express the functional dependence between the aircraft model parameters (the dimensional stability and control derivatives) and the payload mass change as LFTs. Using LFT algebra all of the uncertainties are then combined into one block, and its dimension is reduced to obtain a linear model expressed as an LFT with an uncertainty block of a minimum possible dimension.

A change in the payload mass δ_{pl} causes a change in the total mass of the aircraft

$$m = \hat{m} + \delta_{\mathrm{pl}}$$

where $\hat{m}$ is the total mass of the aircraft with a nominal payload. This equation can be expressed as an LFT:

$$m = \mathcal{F}_l\left(\begin{bmatrix} \hat{m} & 1 \\ 1 & 0 \end{bmatrix}, \delta_{\mathrm{pl}}\right) \tag{11}$$

The change δ_{pl} affects also the location of the c.g. of the total aircraft/payload system. The new c.g. location is given by

$$x_{\mathrm{cg}} = \hat{x}_{\mathrm{cg}} + \delta_{\mathrm{cg}} \tag{12}$$

where the perturbed and the nominal c.g. locations x_{cg} and $\hat{x}_{cg}$, respectively, are measured from some known reference point. The change δ_{cg} is related to the payload mass variation δ_{pl} by

$$\delta_{cg} = \frac{x_{pl} - \hat{x}_{cg}}{\hat{m} + \delta_{pl}} \delta_{pl} \tag{13}$$

Here x_{pl} is the hanging point of the payload, placed at known location along the aircraft x axis, while y_{pl} and z_{pl} are negligible. The payload is assumed to be a point mass. Equation (13) can be written as an LFT

$$\delta_{cg} = \mathcal{F}_l\left(\begin{bmatrix} 0 & \bar{x}_{pl} \\ \dfrac{1}{\hat{m}} & -\dfrac{1}{\hat{m}} \end{bmatrix}, \delta_{pl}\right) \tag{14}$$

where $\bar{x}_{pl} \triangleq x_{pl} - \hat{x}_{cg}$.

The aircraft moment of inertia I_{yy} also depends on the payload mass and is dominantly affected by the change in x_{cg}. Since the entire range of x_{cg} in the example of this work is only about 5 cm, however, its effect on I_{yy} is less than 2% and, thus, is neglected.

The linear model of the RPV/payload system and its dependence on the δ_{pl} uncertainty is obtained by the following steps: first the aerodynamic model is discussed; then the aircraft short period model is constructed around the unknown c.g. location, and the model parameters are expressed as LFTs of the change in the payload mass; finally, the δ_{pl} are pulled out into a single uncertainty block, and the dimension of the resulting LFT is reduced to minimum, as described in the preceding section.

A. Aerodynamic Model

The aerodynamic forces and moments are expressed relative to the fixed nominal aircraft/payload c.g. Lift, drag, and the aerodynamic pitch moment are given by

$$L = q_d S C_L, \qquad D = q_d S C_D \qquad M = q_d S c C_M \tag{15}$$

where $q_d \triangleq \frac{1}{2}\rho U^2$ is the dynamic pressure, S a reference area (the wing area), c a reference length (the average cord), and U the flight velocity. The lift and pitch moment coefficients C_L and C_M, respectively, are given by the linear models

$$C_L = C_{L_0} + C_{L_\alpha}\hat{\alpha} + C_{L_q}(c/2U_0)q + C_{L_{\delta_e}}\delta_e \tag{16}$$

$$C_M = C_{M_0} + C_{M_\alpha}\hat{\alpha} + \frac{c\left(C_{M_{\dot{\alpha}}}\dot{\hat{\alpha}} + C_{M_q}q\right)}{2U_0} + C_{M_{\delta_e}}\delta_e \tag{17}$$

whereas the drag coefficient C_D is expressed as

$$C_D = C_{D_0} + K C_L^2 \tag{18}$$

where $\hat{\alpha}$ is the angle of attack of the aircraft at the nominal c.g. and will be discussed in the next subsection. U_0 is the trim velocity and δ_e is the elevator deflection.

It is assumed that a change in the payload mass does not change the shape and, thus, the aerodynamic properties of the aircraft. In particular, the aerodynamic moment derivatives are unchanged as long as those are computed relative to a fixed point that in this work is the nominal c.g. location. A change in the payload mass does, though, affect the trim angle of attack α_{trim}, and, thus, varies the nondimensional aerodynamic derivatives on the right-hand side of Eqs. (16–18). In the example examined in this paper and for the payload mass variations considered, α_{trim} varies between 0 and 2 deg only, which is clearly in the range of linear dependence of the non-dimensional aerodynamic coefficients on the dynamic and control variables. Thus, it is assumed that the aerodynamic derivatives in Eqs. (16–18) do not vary with the change in the payload mass.

B. Short Period Model

The aircraft equations of motion are expressed in a body-fixed coordinate system, with the x and z axes pointing forward and down, respectively. The origin of this coordinate system can be chosen either at the known c.g. of the aircraft loaded with a nominal payload or at the unknown true c.g. In this work, the latter option was chosen, resulting in a model that better fits the available robust control design techniques.

Two perturbation variables are used to express the short-period dynamics of the aircraft around the straight and level trim flight: the vertical velocity of the true c.g. w and the pitch rate q. The linearized dynamic equations of motion, the vertical force and the pitch moment equations, are given by[1]

$$\dot{w}(t) = U_0 q - (1/m)\,\Delta Z \tag{19a}$$

$$\dot{q}(t) = (1/I_{yy})\,\Delta M \tag{19b}$$

ΔZ and ΔM are small perturbations of the vertical force $\mathcal{Z}$ and the pitch moment around the true c.g. $\mathcal{M}$, respectively, given by

$$\mathcal{Z} = -L\cos\hat{\alpha} - D\sin\hat{\alpha} \tag{20a}$$

$$\mathcal{M} = M + \mathcal{Z}\delta_{cg} \tag{20b}$$

The angle of attack $\hat{\alpha}$ at the nominal c.g. is related to the velocity components of the true c.g. and the pitch rate by

$$\hat{\alpha} \approx \frac{(w + q\delta_{cg})}{U_0} \approx \alpha + \frac{q}{U_0}\delta_{cg} \tag{21}$$

where $\alpha \approx w/U_0$ is the angle of attack at the true c.g. location.

The perturbations ΔZ and ΔM are expressed as linear functions of the motion and control variables by substituting $\hat{\alpha}$ of Eq. (21) into Eqs. (15) and (20), differentiating, and substituting the trim condition of straight and level flight. The result is

$$\Delta Z/m \approx Z_w w + Z_q q + Z_{\delta_e}\delta_e \tag{22a}$$

$$\Delta M/I_{yy} \approx M_w w + M_{\dot{w}}\dot{w} + M_q q + M_{\delta_e}\delta_e \tag{22b}$$

where the force and pitch moment dimensional derivatives are defined as

$$Z_{(\cdot)} \triangleq \frac{1}{m}\frac{\partial \mathcal{Z}}{\partial(\cdot)}, \qquad M_{(\cdot)} \triangleq \frac{1}{I_{yy}}\frac{\partial \mathcal{M}}{\partial(\cdot)} \tag{23}$$

and are expressed as functions of the nondimensional aerodynamic derivatives by

$$Z_w = -(q_d S/mU_0)\left(C_{L_\alpha} + C_D\right) \tag{24a}$$

$$Z_q = -(q_d S/m)\,(c/2U_0)\left[C_{L_q} + (2/c)\left(C_{L_\alpha} + C_D\right)\delta_{cg}\right] \tag{24b}$$

$$Z_{\delta_e} = -(q_d S/m)\,C_{L_{\delta_e}} \tag{24c}$$

$$M_w = \frac{q_d S c}{I_{yy}U_0}\left(C_{M_\alpha} - \frac{C_{L_\alpha} + C_D}{c}\delta_{cg}\right) \tag{24d}$$

$$M_{\dot{w}} = (q_d S c/I_{yy})\,(c/2U_0)C_{M_{\dot{\alpha}}} \tag{24e}$$

$$M_q = \frac{q_d S c}{I_{yy}}\frac{c}{2U_0}\left[C_{M_q} + \frac{2C_{M_\alpha} - C_{L_q}}{c}\delta_{cg} - \frac{2}{c^2}\left(C_{L_\alpha} + C_D\right)\delta_{cg}^2\right] \tag{24f}$$

$$M_{\delta_e} = (q_d S c/I_{yy})\left[C_{M_{\delta_e}} - \left(C_{L_{\delta_e}}/c\right)\delta_{cg}\right] \tag{24g}$$

These dimensional derivatives can also be expressed as LFTs of δ_{pl} by substituting the LFT expressions for the total mass m and the c.g. shift δ_{cg} given in Eqs. (11) and (14), respectively, into Eqs. (24) and simplifying the results by the procedure described in the preceding section. As stated before, the dependence of I_{yy} on δ_{pl}

is neglected and, thus, $M_{\dot w}$ given in Eq. (24e) is not a function of δ_{pl}. The other dimensional derivatives expressed as LFTs are given by

$$Z_w = \mathcal{F}_l\left(\begin{bmatrix} \hat{Z}_w & 1 \\ -\dfrac{\hat{Z}_w}{\hat{m}} & -\dfrac{1}{\hat{m}} \end{bmatrix}, \delta_{\text{pl}}\right) \tag{25a}$$

$$Z_q = \mathcal{F}_l\left(\left[\begin{array}{c:cc} \hat{Z}_q & \dfrac{1}{\hat{m}} & \dfrac{\bar{x}_{\text{pl}}}{\hat{m}} \\ \hdashline -\hat{Z}_q & -\dfrac{1}{\hat{m}} & -\dfrac{\bar{x}_{\text{pl}}}{\hat{m}} \\ \hat{Z}_w & 0 & -\dfrac{1}{\hat{m}} \end{array}\right], \begin{bmatrix} \delta_{\text{pl}} & 0 \\ 0 & \delta_{\text{pl}} \end{bmatrix}\right) \tag{25b}$$

$$Z_{\delta_e} = \mathcal{F}_l\left(\begin{bmatrix} \hat{Z}_{\delta_e} & 1 \\ -\dfrac{\hat{Z}_{\delta_e}}{\hat{m}} & -\dfrac{1}{\hat{m}} \end{bmatrix}, \delta_{\text{pl}}\right) \tag{25c}$$

$$M_w = \mathcal{F}_l\left(\begin{bmatrix} \hat{M}_w & 1 \\ \dfrac{\hat{Z}_w}{I_{yy}}\bar{x}_{\text{pl}} & -\dfrac{1}{\hat{m}} \end{bmatrix}, \delta_{\text{pl}}\right) \tag{25d}$$

$$M_q = \mathcal{F}_l\left(\left[\begin{array}{c:cc} \hat{M}_q & \bar{x}_{\text{pl}} & 0 \\ \hdashline \dfrac{\hat{Z}_q}{I_{yy}} + \dfrac{\hat{M}_w}{\hat{m}} & -\dfrac{1}{\hat{m}} & \dfrac{\bar{x}_{\text{pl}}}{\hat{m}} \\ \dfrac{\hat{Z}_w}{I_{yy}} & 0 & -\dfrac{1}{\hat{m}} \end{array}\right], \begin{bmatrix} \delta_{\text{pl}} & 0 \\ 0 & \delta_{\text{pl}} \end{bmatrix}\right) \tag{25e}$$

$$M_{\delta_e} = \mathcal{F}_l\left(\begin{bmatrix} \hat{M}_{\delta_e} & 1 \\ \dfrac{\hat{Z}_{\delta_e}}{I_{yy}}\bar{x}_{\text{pl}} & -\dfrac{1}{\hat{m}} \end{bmatrix}, \delta_{\text{pl}}\right) \tag{25f}$$

where $\hat{Z}_{(\cdot)}$ and $\hat{M}_{(\cdot)}$ are the nominal dimensional derivatives and can be obtained by substituting $\delta_{\text{cg}} = 0$ and $m = \hat{m}$ into Eqs. (24).

Using the linearized model for the force and moment perturbations given in Eqs. (22), the short-period equations of motion (19) expressed as a transfer matrix become

$$\begin{Bmatrix} w(s) \\ q(s) \end{Bmatrix} = \left[\begin{array}{cc|c} Z_w & Z_q + U_0 & Z_{\delta_e} \\ \bar{M}_w & \bar{M}_q & \bar{M}_{\delta_e} \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \delta_e(s) \tag{26}$$

where the modified moment derivatives $\bar{M}_{(\cdot)}$ are

$$\bar{M}_w \triangleq M_w + M_{\dot w} Z_w, \qquad \bar{M}_{\delta_e} \triangleq M_{\delta_e} + M_{\dot w} Z_{\delta_e}$$
$$\bar{M}_q \triangleq M_q + M_{\dot w}(Z_q + U_0) \tag{27}$$

As in Eq. (6), the transfer matrix in Eq. (26) can be expressed as an LFT:

$$\begin{Bmatrix} w(s) \\ q(s) \end{Bmatrix} = \mathcal{F}_u\left(\mathcal{P}_{\text{sp}}, \frac{1}{s} I_2\right) \delta_e(s) \tag{28}$$

where the system matrix $\mathcal{P}_{\text{sp}}$ is defined as

$$\mathcal{P}_{\text{sp}} \triangleq \left[\begin{array}{cc:c} Z_w & Z_q + U_0 & Z_{\delta_e} \\ \bar{M}_w & \bar{M}_q & \bar{M}_{\delta_e} \\ \hdashline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \tag{29}$$

Substituting the dimensional derivatives expressed as LFTs of δ_{pl} in Eqs. (25) into Eqs. (29) and (27), pulling out all of the δ_{pl} and minimizing the dimension of the uncertainty block results in a short-period model, the system matrix of which is expressed as an LFT:

$$\mathcal{P}'_{\text{sp}} = \mathcal{F}_l(M'_{\text{sp}}, \delta_{\text{pl}} I_3) \tag{30a}$$

where I_3 is a 3×3 identity matrix. M'_{sp} is given by

$$M'_{\text{sp}} = \left[\begin{array}{ccc:ccc} \hat{Z}_w & \hat{Z}_q + U_0 & \hat{Z}_{\delta_e} & 1 & \dfrac{\bar{x}_{\text{pl}}}{\hat{m}} & 0 \\ \hat{\bar{M}}_w & \hat{\bar{M}}_q & \hat{\bar{M}}_{\delta_e} & C_0\hat{m} & C_0\bar{x}_{\text{pl}} & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline -\dfrac{\hat{Z}_w}{\hat{m}} & -\dfrac{\hat{Z}_q}{\hat{m}} & -\dfrac{\hat{Z}_{\delta_e}}{\hat{m}} & -\dfrac{1}{\hat{m}} & -\dfrac{\bar{x}_{\text{pl}}}{\hat{m}^2} & 0 \\ 0 & \hat{Z}_w & 0 & 0 & -\dfrac{1}{\hat{m}} & 0 \\ 0 & \dfrac{\hat{M}_w}{\hat{m}}\bar{x}_{\text{pl}} + \dfrac{\hat{Z}_w}{I_{yy}}\bar{x}_{\text{pl}}^2 & 0 & 0 & 0 & -\dfrac{1}{\hat{m}} \end{array}\right] \tag{30b}$$

where

$$C_0 = (\hat{M}_{\dot w}/\hat{m}) - (\bar{x}_{\text{pl}}/I_{yy})$$

Since there is only one uncertain parameter δ_{pl}, the procedure of Sec. III.B assures a minimal-order LFT.

Here, $\bar{x}_{\text{pl}}/\hat{m}^2$, the (5, 5) entry of M'_{sp}, is often very small. If this term is neglected, the LFT can be further reduced to obtain

$$\mathcal{P}_{\text{sp}} = \mathcal{F}_l(M_{\text{sp}}, \delta_{\text{pl}} I_2) \tag{31a}$$

$$M_{\text{sp}} = \left[\begin{array}{ccc:cc} \hat{Z}_w & \hat{Z}_q + U_0 & \hat{Z}_{\delta_e} & 1 & 0 \\ \hat{\bar{M}}_w & \hat{\bar{M}}_q & \hat{\bar{M}}_{\delta_e} & \hat{M}_{\dot w} - \dfrac{\hat{m}}{I_{yy}}\bar{x}_{\text{pl}} & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hdashline -\dfrac{\hat{Z}_w}{\hat{m}} & \dfrac{\hat{Z}_w\bar{x}_{\text{pl}} - \hat{Z}_q}{\hat{m}} & -\dfrac{\hat{Z}_{\delta_e}}{\hat{m}} & -\dfrac{1}{\hat{m}} & 0 \\ 0 & \dfrac{\hat{M}_w}{\hat{m}}\bar{x}_{\text{pl}} + \dfrac{\hat{Z}_w}{I_{yy}}\bar{x}_{\text{pl}}^2 & 0 & 0 & -\dfrac{1}{\hat{m}} \end{array}\right] \tag{31b}$$

This model describes exactly the functional dependence of the aircraft short-period dynamics on the mass of the payload and will be referred as the functional dependence (FD) model in the sequel. The uncertainty block is a repeated scalar block, the dimension of which was reduced to two, even though only one physical parameter, the payload mass, is uncertain. This resulted from the nonlinear dependencies of the model parameters on the uncertainty δ_{pl}.

V. μ Controller Design

The FD model derived in the preceding section is used to design a pitch rate control system of an RPV carrying a payload with an unknown mass. The controller is designed for a straight and level trim condition at an altitude of 5000 ft and a nominal velocity of 30 m/s. It is assumed that only the elevator is used for control and that pitch rate measurements only are available. An implicit model

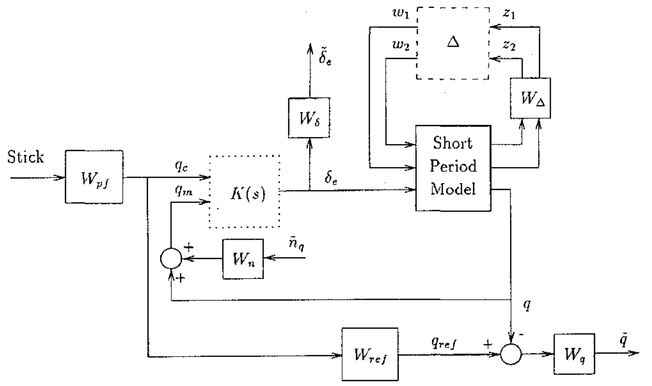

Fig. 5 Pitch rate synthesis interconnection model.

following controller was designed. The reference pitch rate response to a stick input is described by a second-order system[13]:

$$W_{\text{ref}} = K_q \frac{\tau s + 1}{s^2 + 2\zeta_{\text{sp}}\omega_{\text{sp}} s + \omega_{\text{sp}}^2}, \qquad \tau \triangleq \frac{mU_0}{L_\alpha} \tag{32}$$

with damping of $\zeta_{\text{sp}} = 0.7$ and a natural frequency of $\omega_{\text{sp}} = 0.5/\tau$ (Ref. 14). K_q is set to $0.5\,g/U_0$ to produce $0.5\,g$ acceleration for maximum anticipated stick deflection.

The complete interconnection model used for the μ design is shown in Fig. 5. This model includes all of the robustness and the performance weights used in the design. In this example it was assumed that the RPV empty mass is 157 kg and the payload mass can vary in the range of 0–60 kg. Thus, the nominal aircraft model in Eqs. (31) is computed for an average payload mass of 30 kg (nominal mass of the system is 187 kg) and, consequently, δ_{pl} is in the range of ±30 kg, with $\Delta = \delta_{\text{pl}} I_2$. To normalize the uncertainty block to $\|\Delta\| \leq 1$, the robustness weight W_Δ is set to

$$W_\Delta = \begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix} \tag{33}$$

The error between the aircraft and the reference model responses, $\tilde{q}$, is required to be less than 10 mrad/s at all frequencies. Thus, the performance weight W_q is constant and set to 1/0.01. The pilot stick input is prefiltered by

$$W_{pf} = \frac{(s/100) + 1}{(s/8) + 1} \tag{34}$$

to model band-limited pilot inputs.

The RPV elevator has a maximum deflection of $\delta_e^{\max} = 11$ deg. High-frequency elevator commands should be avoided since the servo dynamics have a bandwidth of about 22 rad/s. Thus, the weight on the elevator command is chosen as

$$W_\delta = \frac{1}{\delta_e^{\max}} \frac{(s/20) + 1}{(s/50) + 1} \tag{35}$$

In addition, it is assumed that the pitch rate measurements include a white, zero mean measurement noise with a spectral density of $0.008\sqrt{(\text{rad/s})}$. To normalize this input signal, W_n was set to 0.008.

A μ controller was designed using Refs. 8 and 9. The design process converged after three D–K iterations and achieved a μ of 0.973 with a 18th-order controller. The order of this controller was reduced using the optimal Hankel norm approximation technique[15] to order 5, while keeping the value of μ under 0.985.

The closed-loop system response of the RPV carrying a payload with nominal, maximum and minimum mass while using the reduced-order controller are shown in Fig. 6. The stick input was a doublet of maximum amplitude. The RPV pitch rate q, the error between the reference model and the aircraft pitch rate $\tilde{q}$, and the elevator deflection δ_e are presented. The good robust performance characteristics of the system obtained using the FD-based μ controller are clearly evident from these figures.

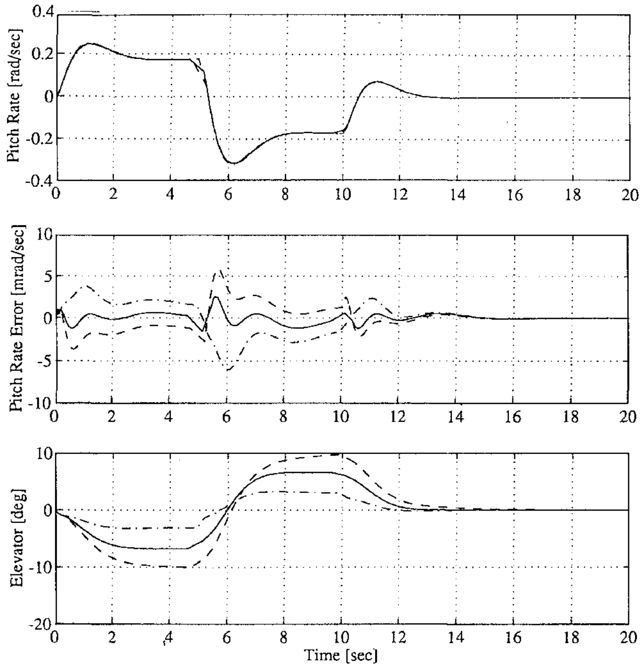

Fig. 6 Closed-loop response of the RPV with a 5th-order μ controller based on the functional dependence model: ——, payload = nominal; – – –, payload = nominal + 30 kg; and ·–·–·, payload = nominal −30 kg.

VI. Controller Comparison

The controller of the preceding section, referred in the sequel as the μ_{FD} controller, is compared here to a μ controller that was designed using the short-period model of Eq. (26) with bounds on each coefficient of the state–space matrices, similar to Ref. 16. This controller will be referred to as the parameter bounding (PB) μ_{PB} controller.

For the μ_{PB} controller design, the maximum perturbation of each coefficient in the short-period model from its nominal value is computed for the anticipated payload mass range. The perturbation in Z_q is neglected since it is small compared to the nominal $\hat{Z}_q + U_0$ (U_0 is dominant in this expression). An LFT with five independent δ is constructed, which expresses the possible model perturbations. The interconnection model and the weighting functions used for this design are as presented in the preceding section.

The μ design process converged after seven D–K iterations to a value of $\mu = 1.368$ with a 26th-order controller. This μ value indicates that robust performance cannot be obtained for all possible plant perturbations considered by the PB uncertainty model, in which the model parameters vary independently in a range symmetric about their nominal values. Because of the functional dependence of the model parameters, this model is overbounded. The worst perturbation may not be physical, as in fact is the case in this example. Figure 7 shows the worst-case parameter perturbations of the μ_{PB} controller. It can be seen that the worst case (marked by circles on the figures) is a coefficient combination that cannot result from RPV mass variations.

To obtain robust performance with this model, the uncertainty bounds must be reduced and/or the performance requirements have to be relieved. The result is a controller that either does not guarantee robust performance for the entire payload mass range or provides only inferior performance for the whole uncertainty range, respectively. By trial and error it was found that robust performance can be achieved by reducing the mass variation from ±30 kg to about ±12 kg [see Eq. (33)], with a 27th-order controller and $\mu_{\text{PB}} = 0.999$, which was reduced by optimal Hankel norm approximation to order 9 without affecting the μ value. In this case too, the worst-case parameter perturbations are not physical, as can be seen in Fig. 7, marked by crosses. Alternatively, when relieving the performance requirements, again by trail and error, the performance specification had to be reduced 16-fold(!) to obtain a 29th-order controller with $\mu_{\text{PB}} = 0.985$.

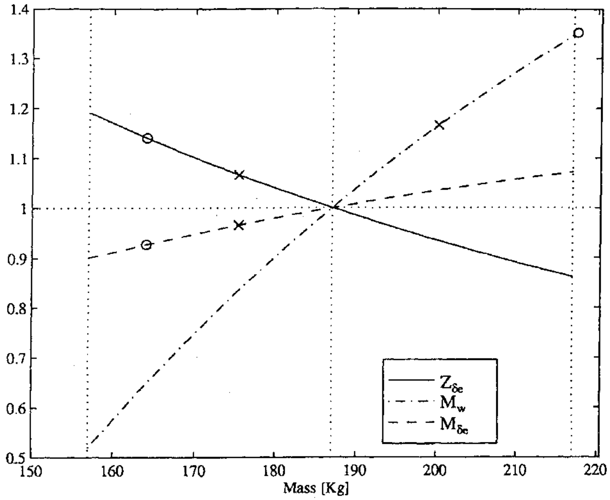

Fig. 7 PB model coefficients as a function of the RPV mass with each coefficient normalized by its nominal value; worst perturbations of the PB μ controller marked by circles for the full payload mass range design and by crosses for the reduced payload mass range case.

A designer using the PB uncertainty model for this case is required to either relax the design requirements on the performance, or design several controllers for a few payload mass ranges and establish a switching scheme between them. Another option is to check by simulation several mass points and consider the performance obtained. This approach is relatively simple in this example with a single independent uncertainty variable, but becomes complicated as more uncertainty variables are introduced. In addition, such evaluation does not guarantee performance for parameter values not checked. Lacking the FD model prohibits the analysis of the true effect of the uncertainty variables on the design.

The μ_{PB} controller obtained for the full mass range and performance requirements was analyzed using the FD model, yielding a μ_{FD} value of 1.144. This implies that when considering only those perturbations possible physically, i.e., removing the overbounding in the analysis, the controller performs better than is deduced from the μ_{PB} value of 1.368, but still falls short of providing the required robust performance. On the other hand, the μ controller designed using the FD model achieved the robust performance specifications by considering physical perturbations only.

From this comparison it is clear that the μ_{FD} controller has superior characteristics. Thus, the use of the accurate uncertainty model can be beneficial in achieving a better compromise between performance and robustness goals.

VII. Summary and Conclusions

In this paper an approach for modeling linear dynamic systems, the parameters of which are dominated by a few unknown physical parameters, was presented. The modeling strategy is based on incorporating the known functional dependence between the model parameters and the physical quantities into the uncertainty model using LFT arithmetics. The obtained model accurately describes the system uncertainty without undesirable uncertainty space overbounding. Such modeling may also result in an uncertainty block of smaller dimension, which is beneficial for the controller design by reducing its dimension and complexity. The modeling scheme may also be exploited when designing for several operating points distinguished by different values of physical parameters. By considering these physical parameters as uncertainties of the model, a single controller for all operating points can be synthesized.

In the case where the functional dependence of the model parameters on the physical quantities is complex or unknown and can be obtained from experiments only, the dependence can be derived empirically and incorporated in a similar fashion. Such applications are the subject of future research.

As an example, the short-period dynamics of an unmanned flight vehicle carrying a payload with an unknown mass are modeled using the proposed methodology. The linear model obtained is expressed as an LFT, where the uncertainty block is a two-by-two repeated scalar block with the payload mass as the uncertainty. This model was then used to design an implicit model following μ controller that satisfied all of the robust performance requirements. The performance of this controller was compared to a standard μ controller, designed based on a model with independent uncertainties on the state-space parameters. The latter was able to provide robust performance only for a reduced mass range with a higher order controller. These inferior characteristics are because of the overbounding of the actual uncertainty space.

In summary, from the example it is evident that by including the functional dependence of the model parameters on a few physical parameters it is possible to obtain a less conservative controller with better performance while maintaining robustness.

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